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Kapteyn series arising in radiation problems

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Abstract
In discussing radiation from multiple point charges or magnetic dipoles, moving in circles or ellipses, a variety of Kapteyn series of the second kind arises. Some of the series have been known in closed form for a hundred years or more, others appear not to be available to analytic persuasion. This paper shows how 12 such generic series can be developed to produce either closed analytic expressions or integrals that are not analytically tractable. In addition, the method presented here may be of benefit when one has other Kapteyn series of the second kind to consider, thereby providing an additional reason to consider such series anew.

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1. Introduction

One problem in radiation that was considered of great interest at the beginning of the 20th Century is the following. It is well known that a single point charge, moving uniformly in a circle, radiates. Suppose then that one has N charges equally spaced around a circle and all moving at the same circular speed. Then they too radiate. Now as the number N of charges is increased, all other conditions being held fixed, then the spacing between charges decreases proportional to $1/N$. The limit of this process is a continuous uniform charge distribution moving with constant circular motion, i.e., a steady-state ring current. But it is also well known that such a current formation does *not* radiate. Then the question is: As $N \to \infty$ how does the radiation diminish so that, finally, there is no radiation from a continuous ring current?

Investigations of this basic problem immediately encountered Kapteyn series of the second kind (see, e.g., [1, 2]) in a variety of forms and guises. (In general, Kapteyn series of the first kind are infinite sums of Bessel functions of the form

$$F(b) = \sum_{n=1}^{\infty} f_n J_n(nb) \tag{1}$$

and Kapteyn series of the second kind involve two Bessel functions.)

While the formula describing the radiation output was expressible as a set of terms involving sums of Kapteyn series, at first only approximations to the series could be obtained for arbitrary N [3]. The work of Budden [4] provided a systematic determination of the Kapteyn series involved and evaluated the radiation field of the N like particles in terms of factors summed to $N/2 - 1$. The advantage was that, along the way, Budden managed to effect solutions in closed analytical form to some of the Kapteyn series involved. The upshot was that, as $N \to \infty$, one could show how the radiation field diminished to zero.

Since that time there has been, and continues to be, interest in a variety of such radiation types of problems. Alternating positive and negative point charges spread uniformly around a ring, each of which moves at a constant circular speed, are one such problem [5]. As the number of charges increases without limit the spacing between successive charges tends to zero so that, in the limit, there is a charge neutral ring that does not radiate. The approach of the radiation field to zero as the number of charges tends to infinity is the problem of interest. Fortunately, this problem is just a variant of the problem solved by Budden [4] because it represents two rings of opposite charges with twice the spacing. Budden's solution is then immediately appropriate by superposition and charge reversal.

Radiation from a magnetic dipole, off-center from a pulsar that spins, is another such problem [6, 7], as is the radiation field from a charged particle undergoing elliptical motion [8].

In all such problems there, to date, 12 basic Kapteyn series of the second kind have arisen, some of which have been known in closed form for a while others are often referred to as 'solved' but seem to be not readily available, if at all.

This paper provides the basic methodology to handle all 12 of the series and shows which are expressible in closed analytic form and which are expressible only as integrals that cannot be reduced to analytic form.

2. Manipulations with basic sets of Kapteyn series

2.1. The sets of series

The 12 series in question are given by

$$S_1(\lambda, m, b) = \sum_{n=1}^{\infty} \lambda^n n^{2m} J_n^2(nb) \tag{2a}$$

$$S_2(\lambda, m, b) = \sum_{n=1}^{\infty} \lambda^n n^{2m+1} J_n^2(nb) \tag{2b}$$

$$S_3(\lambda, m, b) = \sum_{n=1}^{\infty} \lambda^n n^{2m} J_n'^2(nb) \tag{2c}$$

$$S_4(\lambda, m, b) = \sum_{n=1}^{\infty} \lambda^n n^{2m+1} J_n'^2(nb) \tag{2d}$$

$$S_5(\lambda, m, b) = \sum_{n=1}^{\infty} \lambda^n n^{2m} J_n(nb) J_n'(nb) \tag{2e}$$

$$S_6(\lambda, m, b) = \sum_{n=1}^{\infty} \lambda^n n^{2m+1} J_n(nb) J_n'(nb), \tag{2f}$$

where $\lambda \in \{\pm 1\}$ and $m \in \mathbb{Z}$.

The determination of the sets of series can be reduced to the simpler problem of determining only the set of series with $m = 0$ (in the cases of S_1, S_3 and S_6) and the set of series with $m = -1$ (in the cases of S_2, S_4 and S_5).

The reason for these reductions is as follows. One can write

$$2S_6(\lambda, m, b) = \frac{\partial S_1}{\partial b} \tag{3a}$$

$$2S_5(\lambda, m, b) = \frac{\partial S_2}{\partial b} \tag{3b}$$

so that it is sufficient to obtain S_1, S_2, S_3 and S_4.

Note also that

$$\frac{\partial S_3}{\partial b} = 2\sum_{n=1}^{\infty} \lambda^n n^{2m+1} J_n'(nb) J_n''(nb). \tag{4}$$

But, because Bessel's equation (e.g., [9], section 9.1) gives

$$J_n''(nb) = \frac{1}{b^2}\left[\frac{b}{n} J_n'(nb) + (1 - b^2) J_n(nb)\right] \tag{5}$$

one has

$$b^2 \frac{\partial S_3}{\partial b} + 2bS_3 = (1 - b^2)\frac{\partial S_1}{\partial b} \tag{6}$$

so that

$$S_3 = \frac{1}{b^2}\left[(1 - b^2)S_1 + 2\int_0^b \mathrm{d}x\; S_1(x)\right]. \tag{7}$$

Equally

$$S_4 = \frac{1}{b^2}\left[(1 - b^2)S_2 + 2\int_0^b \mathrm{d}x\; S_2(x)\right]. \tag{8}$$

Thus it is sufficient to obtain S_1 and S_2.

One can also use a version of a theorem due to Watson [10], which states that, if

$$g(b) = \sum_{n=1}^{\infty} a_n J_n(nb) \tag{9}$$

is known, where a_n is arbitrary, then

$$f(b) = \sum_{n=1}^{\infty} \frac{a_n}{n^2} J_n(nb) \tag{10}$$

can be obtained from

$$\left(b\frac{\partial}{\partial b}\right)^2 f(b) = (1 - b^2) g(b). \tag{11}$$

Alternatively, if $f(b)$ is known then $g(b)$ is given by direct differentiation.

Consider then S_1. Use the fact that (e.g., [11], section 6.681)

$$J_n^2(nb) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{d}\psi\, J_{2n}(2nb\cos\psi) \tag{12}$$

so that

$$S_1 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{d}\psi \sum_{n=1}^{\infty} \lambda^n n^{2m} J_{2n}(2nb\cos\psi). \tag{13}$$

But the series

$$h_m(b) = \sum_{n=1}^{\infty} \lambda^n n^{2m} J_{2n}(2nb\cos\psi) \tag{14a}$$

$$\equiv \frac{1}{2^{2m}} \sum_{n=1}^{\infty} \lambda^n (2n)^{2m} J_{2n}(2nb\cos\psi) \tag{14b}$$

is precisely of the form required in Watson's theorem, with $a_n = 0$ if n is odd and $a_n = \exp[\mathrm{i}n\pi/2\ln\lambda]n^{2m}$ if n is even, so that

$$(1-b^2)h_m(b) = \left(b\frac{\partial}{\partial b}\right)^2 h_{m-2}(b). \tag{15}$$

Hence, for $m > 0$ all series of the type S_1 can be reduced to the determination of $h_0(b)$ by differentiation. This procedure was used by Lerche and Tautz [7] to evaluate in closed form the two Kapteyn series of the second kind occurring in the dipole radiation problem discussed by Harrison and Tademaru [6] who had obtained only the lowest order expansion result. Equally, for $m < 0$ one can use Watson's theorem in the converse sense to note that

$$\left(b\frac{\partial}{\partial b}\right)^2 h_{-|m|}(b) = (1-b^2)h_{-|m|+2}(b) \tag{16}$$

so that, by two integrations, one has a recursive relation leading directly to h_0.

Thus, all 12 of the basic series needed can be written in terms of four fundamental series

$$F = \sum_{n=1}^{\infty} \frac{\lambda^n}{n} J_n^2(nb) \tag{17a}$$

$$G = \sum_{n=1}^{\infty} \lambda^n J_n^2(nb) \tag{17b}$$

for $\lambda \in \{\pm 1\}$. All other series (with $m \neq 0$, or $m \neq -1$, respectively) are directly given as simple differentials or simple integrals with respect to b of one or the other of the four fundamental series. It is, therefore, both necessary and sufficient to consider F and G.

2.2. The two series represented by F

Set

$$F_+ = \sum_{n=1}^{\infty} \frac{J_n^2(nb)}{n} \tag{18a}$$

$$F_- = \sum_{n=1}^{\infty} (-1)^n \frac{J_n^2(nb)}{n}. \tag{18b}$$

Now, in F_+, replace the Bessel functions as

$$J_n^2(nb) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{d}\psi\, J_{2n}(2nb\cos\psi) \tag{19}$$

while in F_- replace $(-1)^n J_n^2(nb) = J_n(nb)J_{-n}(nb)$ and

$$J_n(nb)J_{-n}(nb) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{d}\psi\, J_0(2nb\cos\psi)\cos 2n\psi. \tag{20}$$

Then, write

$$J_{2n}(2nb\cos\psi) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\, \cos(2nb\cos\psi\,\sin\theta)\cos 2n\theta \tag{21}$$

and

$$J_0(2nb\cos\psi) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\, \cos(2nb\cos\psi\,\sin\theta). \tag{22}$$

In principle, one could also use a representation of the Bessel function in exponential form (see [11]) and then carry out the summation. However, because equations (18*a*) and (18*b*) are a product of two Bessel functions, this ansatz would be even more difficult than the approach followed here.

Now, inserting equations (19) and (21) into expression (18*a*) for F_+ and inserting equations (20) and (22) into expression (18*b*) for F_- and then performing directly the infinite sums leads, after some tedious but elementary algebra, to

$$F_+ = -\frac{1}{\pi^2}\int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\,\ln\left[\frac{\sin^2(\theta - b\cos\phi\sin\theta)\sin^2(\theta + b\cos\phi\sin\theta)}{\sin^4\theta}\right] \tag{23a}$$

and

$$F_- = -\frac{1}{\pi^2}\int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\,\ln\left[\frac{\cos^2(\theta - b\cos\phi\sin\theta)\cos^2(\theta + b\cos\phi\sin\theta)}{\cos^4\theta}\right] \tag{23b}$$

$$= -\frac{1}{\pi^2}\int_0^{\frac{\pi}{2}}\mathrm{d}\phi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\,\ln\left[\frac{\sin^2(\theta - b\cos\phi\cos\theta)\sin^2(\theta + b\cos\phi\cos\theta)}{\sin^4\theta}\right]. \tag{23c}$$

Numerical investigation by direct summation of F_+ and F_- as given in equations (18*a*), (18*b*) and comparison with the simple integral formulations given in equations (23*a*), (23*c*) show that the series are indeed given by equations (23*a*), (23*c*) to better than a part in 10^4; this limit on resolution being caused by numerical round-off error. Figures 1 and 2 show the comparison between the integrals and direct summation as a function of increasing $b \in]0, 1[$ for both F_+ and F_-, respectively, with the relative error (in %) also being plotted[3].

Throughout the paper, the numerical evaluation of infinite sums is carried out as follows: first, a number of terms (usually 1000) are summed directly; to accelerate the convergence of the sum, then Wynn's epsilon method (see, e.g., [12, 13]) is used, which samples a number of additional terms (usually 100) in the sum, and then tries to fit them to a polynomial multiplied by a decaying exponential. Thus, the series are well approximated and the required computer time is kept moderate. The convergence of the sums, in addition, is guaranteed by analytical considerations. Furthermore, numerical integrations are carried out using standard techniques such as adaptive grids. However, some care has to be taken of the square-root singularity [e.g., at $\phi = \theta = 0$ in equations (23*a*) and (23*c*)]. Since we used Mathematica® version 6.0, this

[3] Note that, for numerical reasons, the relative error increases above 10^{-4}% as $b \to 1$ (figure 1) and as $b \to 0$ (figure 2), respectively. This depends heavily on the numerical summation and integration methods as well as on the computer times. By expansion of the integrals around $b = 1$ and $b = 0$, however, one can get almost exact agreement of the series and the integral.

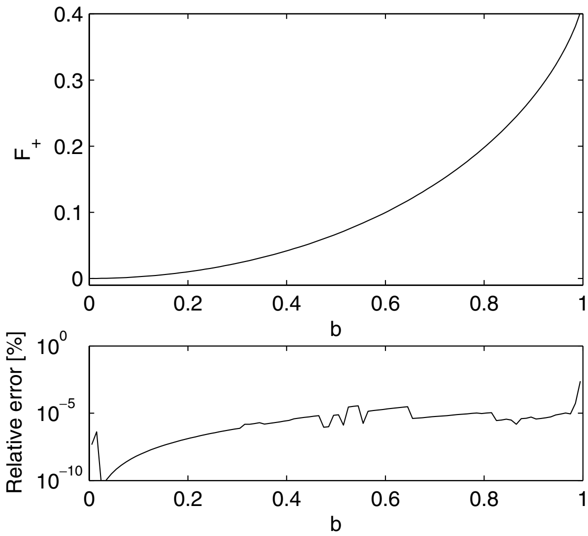

Figure 1. The series F_+ from equation (18*a*) with the relative error when compared to the integral from equation (23*a*).

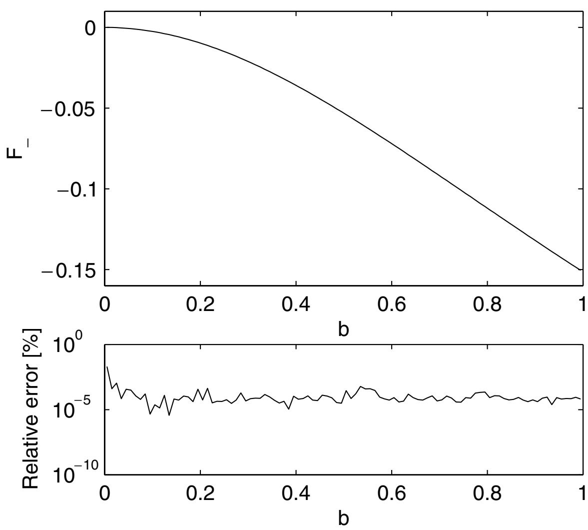

Figure 2. The series F_- from equation (18*b*) with the relative error when compared to the integral from equation (23*c*).

problem is dealt with automatically. Using other packages, however, appropriate measures would have to be taken manually.

Marshall [8] suggested that the sum $F_{\mathrm{M}} \equiv \frac{1}{2}\partial F_+/\partial b$, written in the form

$$F_{\mathrm{M}} = \sum_{n=1}^{\infty} J_n(nb) J_n'(nb), \tag{24}$$

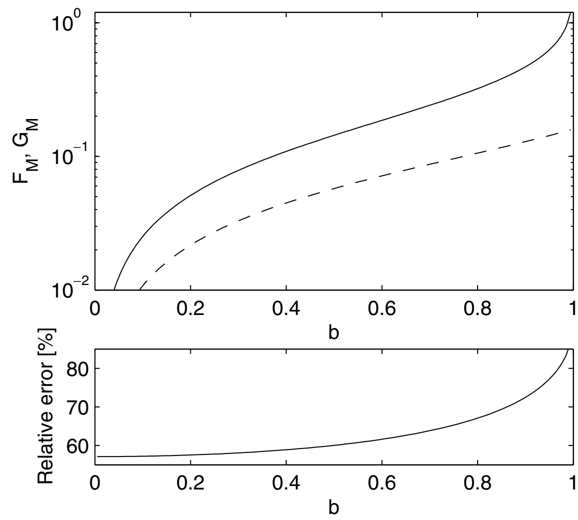

Figure 3. The series F_{M} from equation (24) (solid line) compared to the integral representation G_{M} from equation (25), as given in [8] (dashed line). In the lower panel, the relative error with respect to the direct summation of the series is shown.

could be represented by a single elliptic integral [his equation (23)] as

$$G_{\mathrm{M}} = \frac{1}{\pi b}\int_1^\infty \mathrm{d}u\left(\frac{u}{\sqrt{u^2 - b^2\sin^2 u}} - 1\right). \tag{25}$$

Figure 3 shows plots (as a function of b) of both the sum F_+ and the elliptic integral representation suggested in [8]. There is no agreement even at the crudest level of approximation indicating that the elliptic integral is not appropriate.

2.3. The two series represented by G

Set

$$G_+ = \sum_{n=1}^\infty J_n^2(nb) \tag{26a}$$

$$G_- = \sum_{n=1}^\infty (-1)^n J_n^2(nb). \tag{26b}$$

The series G_+ has been known in closed form since the time of Schott [5]. Use the well-known fact [1] that

$$\frac{1}{1 - b\cos\phi} = 1 + 2\sum_{n=1}^\infty J_n(nb)\cos[n(\phi - b\sin\phi)]. \tag{27}$$

Integrate equation (27) over $0 \leqslant \phi \leqslant \pi$, thereby obtaining

$$\sum_{n=1}^\infty J_n(nb)^2 = \frac{1}{2}\left(\frac{1}{\sqrt{1-b^2}} - 1\right) \tag{28}$$

which is just Schott's [5] formula.

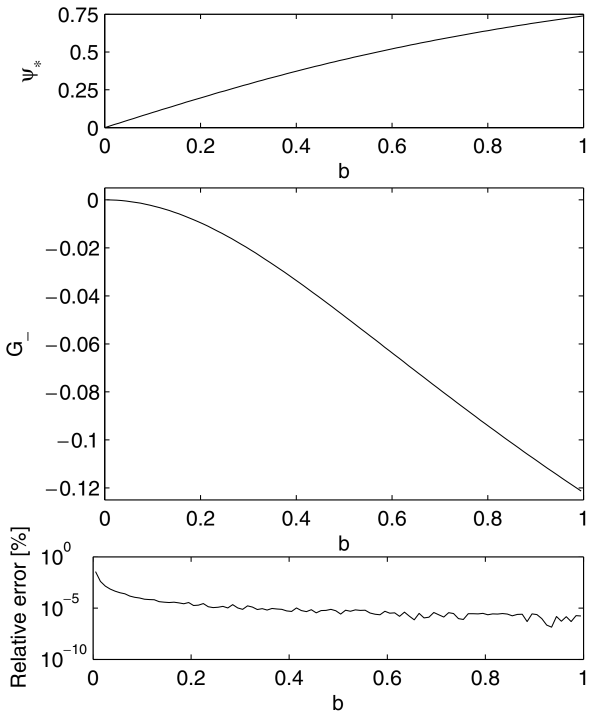

Figure 4. The values for $\psi_\star$ as a function of b (upper panel) and the series G_- from equation (26b) together with the relative error when compared to the integral from equation (32a) (middle and lower panels).

The series G_- is considerably more complicated to evaluate. Write

$$G_- \equiv \sum_{n=1}^{\infty} J_n(nb) J_{-n}(nb)$$
$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{d}\psi \sum_{n=1}^{\infty} J_0(2nb\cos\psi)\cos 2n\psi. \tag{29}$$

Now use the Schlömilch [14] formula

$$f(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{d}\phi\, F(x\sin\phi)$$
$$= \frac{1}{\pi}\int_0^{\pi} \mathrm{d}u\, F(u) + \frac{2}{\pi}\int_0^{\pi} \mathrm{d}u\, F(u)\sum_{n=1}^{\infty} J_0(nx)\cos nu \tag{30a}$$

and set $F(u) = \delta(u - w)$ (in $0 \leqslant w \leqslant \pi$) so that

$$\sum_{n=1}^{\infty} J_0(nx)\cos nw = \frac{1}{2}[\pi f(x) - 1]. \tag{31}$$

With the identifications $w = 2\psi$ and $x = 2b\cos\psi$, equation (29) then yields

$$G_- = -\frac{1}{2} + \frac{1}{\pi}\int_0^{\psi_\star} \frac{\mathrm{d}\psi}{\sqrt{b^2\cos^2\psi - \psi^2}} \tag{32a}$$

where the upper integration limit is implicitly given by $\psi_\star = b\cos\psi_\star$ or b is given explicitly by $b = \psi_\star \sec\psi_\star$. One can then write

$$G_- = -\frac{1}{2} + \frac{\cos\psi_\star}{\pi}\int_0^1 \frac{\mathrm{d}z}{\sqrt{\cos^2(\psi_\star z) - z^2\cos^2\psi_\star}} \tag{32b}$$

which might be more amenable when numerical integration is required. Figure 4 compares G_- given by equation (32*a*) with direct term-by-term summation of the series in equation (26*b*), showing that, to within about 1 part in 10^5, the two are identical in the interval $0 < b < 1$ (cf footnote 3). Note also that the integral representation of G_- is convergent for all values of b, including $b > 1$.

3. Discussion and conclusion

A general method has been presented for the evaluation of 12 Kapteyn series of the second kind. Such series are important for the analytic description of radiation processes in various astrophysical applications such as the radiation from off-centered dipoles in neutron stars. Originally, the Kapteyn series described here arose when the attempt was made to describe the radiation from a distribution of a finite number of discrete point charges, all moving at uniform spacing at a constant speed in a circle.

Previously, most of the Kapteyn series have not been evaluated or, in the case of one of the series, were written in terms of a single elliptic integral, which turned out to be invalid when evaluated numerically (see equation (25)). Instead, a new equation (32*a*) was found, which represents the series G_m in terms of a different, but also elliptic, integral.

As has been shown here by recurrence relations, there are only four basic series that need to be calculated, one of which was already known in closed algebraic form. All other of the 12 series can be obtained from direct differentiation or integration of one or other of the four basic series. The series can be evaluated in terms of closed analytic expressions or in terms of integrals that cannot be further reduced. Numerical calculations were carried out to compare the values obtained by direct summation to those obtained from the integral representations, and the relative errors (less than a part in 10^4) were shown to be limited by numerical round-off errors that are responsible for the differences occurring between direct series representations and integral representations of the series.

Furthermore, the method presented here may be useful when one has other Kapteyn series of the second kind to consider, thereby providing an additional reason to consider such series anew.

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